

# Stochastic Optimal Growth Model with Risk Sensitive Preferences

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**Abstract.** This paper studies a one-sector optimal growth model with i.i.d. productivity shocks that are allowed to be unbounded. The utility function is assumed to be non-negative and unbounded from above. The novel feature in our framework is that the agent has risk sensitive preferences in sense of Hansen and Sargent (1995). Under mild assumptions imposed on the productivity and utility functions we prove that the maximal discounted non-expected utility in the infinite time horizon satisfies the optimality equation and the agent possesses a stationary optimal policy. A new point used in our analysis is an inequality for the so-called associated random variables. We also establish the Euler equation that incorporates the solution to the optimality equation.

**Keywords.** stochastic growth model, entropic risk measure, unbounded utility, unbounded shocks.

## 1. Introduction

This paper deals with one-sector stochastic optimal growth model with possibly unbounded shocks and non-negative utilities that are allowed to be unbounded from above. Unbounded returns are very common in economic models, see Alvarez and Stokey (1998); Boyd (1990); Durán (2000); Le Van and Morhaim (2002) for the deterministic problems, and Durán (2003); Jaśkiewicz and Nowak (2011b); Kamihigashi (2007) for stochastic problems. Most of the aforementioned works apply the weighted supremum norm approach introduced by Wessels (1977). The other group of papers makes use of the idea presented by Rincón-Zapatero and Rodríguez-Palmero (2003) within deterministic framework. Their method rests upon a local contraction and utilises one-sided majorant functions. The extensions of these results to stochastic dynamic programming are reported in Jaśkiewicz and Nowak (2011a); Martins-da-Rocha and Vailakis (2010); Matkowski and Nowak (2011).

The novelty in our model relies on the fact that the agent has risk sensitive preferences of the form

$$V_t = u(a_t) - \frac{\beta}{\gamma} \ln E_t[-\gamma V_{t+1}], \quad (1)$$

where  $\gamma > 0$  is a risk sensitive coefficient,  $\beta \in [0, 1)$  is a time discount factor,  $a_t$  is consumption at time  $t$ ,  $u$  is a felicity function and  $V_t$  is the lifetime utility from period  $t$

onward. Here,  $E_t$  stands for the expectation operator with respect to period  $t$  information. The parameter  $\gamma$  affects consumer's attitude towards risk in future utility. The form of preferences in (1) is due to Hansen and Sargent (1995), who used them to deal with a linear quadratic Gaussian control model. The preferences defined in (1) has several advantages. First of all, they are not time-additive in future utility. Time-additivity, however, requires an agent to be risk neutral in future utility. Risk sensitive preferences, on the other hand, allow the agent to be risk averse in future utility in addition to being risk averse in future consumption. This fact results in partial separation between risk aversion and elasticity of intertemporal substitution.<sup>1</sup> Moreover, as argued by ? risk sensitive preferences are also attractive, because they can be used to model preferences for robustness. It is worth emphasising that risk sensitive preferences of form (1) have found several applications, for instance, in the problems dealing with Pareto optimal allocations (see Anderson (2005)) or small noise expansions (see Anderson et al. (2012)).

Our main results are two-fold. First we establish the optimality equation for the non-expected utility in the infinite time horizon, when the agent has risk sensitive preferences (1). The proof as in the standard expected utility case is based on the Banach contraction principle, see Stokey et al. (1989). However, in order to show that the dynamic programming operator maps a space of certain functions into itself, we have to confine our consideration to concave, non-decreasing and non-negative functions that are bounded in the weighted supremum norm. A novel feature in this analysis is an application of some inequality for the so-called associated random variables. This inequality also plays a crucial role in proving that the a fixed point of dynamic programming operator is indeed the value function. Moreover, it naturally fits into our model, in which the production and utility functions satisfy some mild conditions such as monotonicity and concavity. Here, we would like to emphasise that similarly as in Kamihigashi (2007), we do not assume the Inada conditions for the production function at zero and infinity. Secondly, we establish the Euler equation assuming that the production and utility functions are continuously differentiable. As an by-product, we obtain the existence of a distribution function for the income process governed by the optimal policy. This result owes much previous works by Nishimura and Stachurski (2005), Stachurski (2009) that link the Euler equation with the Foster- Lyapunov functions.

The paper is organised as follows. Section 2 describes the model, risk sensitive preferences of the agent and provides essential assumptions. In Section 3, we use the dynamic programming approach to show that the agent has an optimal stationary policy and the lifetime utility is a solution to the optimality (Bellman) equation. Section 4 establishes the Euler equation. Section 5 makes use of the Euler equation to define a Foster-Lyapunov function, which is applied to the proof of stability of the optimal program. Finally, Section 6 gives two examples that illustrate our theory.

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<sup>1</sup>Consult Epstein and Zin (1989), Tallarini (2000), Miao (2014) for more discussion on this topic. The reader is also referred to Chapters 1 and 3 in Becker and Boyd (1997) that constitute a strong motivation for the study of non-time-additive objective functions.

## 2. The Model

This section contains a formulation of the stochastic optimal growth model with the payoff criterion that in a particular case reduces to the one studied by Brock and Mirman (1972). The symbols  $\mathbb{R}_+$  and  $\mathbb{R}_{++}$  denote the non-negative and positive real numbers, respectively. Let  $\mathbb{N}$  be the set of positive integers. The process evolves as follows. At time  $t \in \mathbb{N}$  the agent has an income  $x_t$ , which is divided between consumption  $a_t$  and investment (saving)  $y_t$ . From consumption  $a_t$  the agent receives utility  $u(a_t)$  (which is independent of  $x_t$ ). Investment is used for production with input  $y_t$  yielding output  $x_{t+1} = f(y_t, \xi_t)$ , where  $(\xi_t)_{t \in \mathbb{N}}$  is a sequence of i.i.d. shocks with distribution  $\nu \in \text{Pr}(\mathbb{R}_+)$  and  $f : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$  is a production function. It is assumed that  $x_1 \in \mathbb{R}_+$  is non-random.

We make the following assumptions.

- (U1) The function  $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is continuous at zero, strictly concave, increasing and  $u(0) = 0$ .
- (U2) There exist a constant  $d > 0$  and a continuous non-decreasing function  $w : \mathbb{R}_+ \mapsto [1, \infty)$  such that

$$u(x) \leq dw(x), \quad \text{for all } x \in \mathbb{R}_+$$

and (2) holds.

- (F1) For every  $z \geq 0$  the function  $f(\cdot, z) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is continuous, concave, non-decreasing and for every  $y \geq 0$  the function  $f(y, \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is Borel measurable.
- (F2) There exists a constant  $\alpha \in (0, 1/\beta)$  such that

$$\sup_{y \in [0, x]} \int_{\mathbb{R}_+} w(f(y, z)) \nu(dz) \leq \alpha w(x), \quad \text{for all } x \in \mathbb{R}_+. \quad (2)$$

Put

$$D := \{(x, y) : x \in \mathbb{R}_+, y \in [0, x]\}.$$

For any  $t \in \mathbb{N}$ , by  $H_t$  we denote the set of all sequences

$$h_t = \begin{cases} x_1, & \text{for } t = 1, \\ (x_1, y_1, x_2, \dots, x_t), & \text{for } t \geq 2, \end{cases}$$

where  $(x_k, y_k) \in D$  for all  $k \in \mathbb{R}_+$ . Hence,  $H_t$  is the set of all feasible histories of the income-investment process up to date  $t$ . An *investment policy*  $\pi$  is a sequence  $(\pi_t)_{t \in \mathbb{N}}$ , where  $\pi_t$  is a measurable mapping which associates any admissible history  $h_t$  with an action  $y_t \in [0, x_t]$ . By  $\Pi$  we denote the set of *all investment policies*. We restrict our attention to non-randomised policies, which are enough to study the discounted optimal growth models. A formal definition of a general policy can be found in Stokey et al. (1989). Let  $\Phi$  be the set of all Borel measurable functions such that  $\phi(x) \in [0, x]$  for every  $x \in \mathbb{R}_+$ . A *stationary (investment) policy* is a constant sequence  $\pi$  with  $\pi_t = \phi$  for every  $t \in \mathbb{N}$ . We shall identify a stationary policy with the member of the sequence, i.e., with the mapping  $\phi$ . By  $\Phi$  we also denote the set of all stationary investment policies.

In this paper, we shall consider the *non-expected utility* of the agent in the infinite time horizon derived with the aid of the so-called entropic risk measure. In order to define this measure let us denote by  $(\Omega, \mathcal{F}, P)$  a probability space and let  $X$  be a non-negative random payoff defined on  $(\Omega, \mathcal{F}, P)$ . The entropic risk measure of  $X$  is

$$\rho(X) = -\frac{1}{\gamma} \ln \int_{\Omega} e^{-\gamma X(\omega)} P(d\omega),$$

where  $\gamma > 0$  is the *risk sensitive coefficient*. Let  $X, Y$  be non-negative random variables on  $(\Omega, \mathcal{F}, P)$ . The following properties of  $\rho$  are important in the analysis (see p. 184 in Föllmer and Schied (2004)):

(P1) monotonicity, i.e., if  $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$

(P2) concavity, i.e.,  $\rho(\lambda X + (1 - \lambda)Y) \geq \lambda \rho(X) + (1 - \lambda)\rho(Y)$  for any  $\lambda \in [0, 1]$ .

However, this measure is not positive homogeneous, i.e.,  $\rho(\alpha X) \neq \alpha \rho(X)$  for  $\alpha \in \mathbb{R}_{++}$ , which does not make our analysis straightforward (see the proofs of Lemma 4 and Theorem 1). Here, we wish only to mention that making use of the Taylor expansions for the exponential and logarithmic functions, we can approximate  $\rho(X)$  as follows

$$\rho(X) \approx EX - \frac{\gamma}{2} \text{Var} X,$$

if  $\gamma > 0$  is sufficiently close to 0. Therefore, if  $X$  is a random payoff, then the agent who evaluates his expected payoff with the aid of the entropic risk measure, thinks not only of the expected value  $EX$  of the random payoff  $X$ , but also of its variance. Further comments on the entropic risk measure can be found in Bäuerle and Rieder (2011); Föllmer and Schied (2004) and references cited therein.

Assume that  $k \in \mathbb{N}$ . We say that a function  $v_k \in B_w(H_k)$ , if  $v_k : H_k \mapsto \mathbb{R}_+$  is Borel measurable and there exists a constant  $d_{v_k} \geq 0$  such that  $v_k(h_k) \leq d_{v_k} w(x_k)$  for every  $h_k \in H_k$ . Here,  $w$  is a function used in (U2) and (F2). Let  $\pi = (\pi_k)_{k \in \mathbb{N}} \in \Pi$  be any policy. For  $v_{k+1} \in B_w(H_{k+1})$  and given  $h_k \in H_k$  we put

$$\rho_{\pi_k, h_k}(v_{k+1}) := -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma v_{k+1}(h_k, \pi_k(h_k), f(\pi_k(h_k), z))} \nu(dz). \quad (3)$$

Observe that by Jensen's inequality and (F2) we have

$$\begin{aligned} 0 &\leq \rho_{\pi_k, h_k}(v_{k+1}) \leq \int_{\mathbb{R}_+} v_{k+1}(h_k, \pi_k(h_k), f(\pi_k(h_k), z)) \nu(dz) \\ &\leq d_{v_{k+1}} \int_{\mathbb{R}_+} w(f(\pi_k(h_k), z)) \nu(dz) \\ &\leq d_{v_{k+1}} \sup_{y \in [0, x_k]} \int_{\mathbb{R}_+} w(f(y, z)) \nu(dz) \leq d_{v_{k+1}} \alpha w(x_k) \end{aligned} \quad (4)$$

for any  $h_k \in H_k$  and  $k \in \mathbb{N}$ . Furthermore, we define the operator  $L_{\pi_k}$  as follows

$$L_{\pi_k} v_{k+1}(h_k) := u(x_k - \pi_k(h_k)) + \beta \rho_{\pi_k, h_k}(v_{k+1}),$$

where  $\beta \in (0, 1)$  is a subjective discount factor. By property (P1), it follows that  $L_{\pi_k}$  is monotone, i.e.,  $L_{\pi_k} v_{k+1}(h_k) \leq L_{\pi_k} \hat{v}_{k+1}(h_k)$  for  $h_k \in H_k$  and  $v_{k+1} \leq \hat{v}_{k+1}$ . Moreover, by (4), (U2) and (F2) we get that

$$0 \leq L_{\pi_k} v_{k+1}(h_k) \leq (d + \alpha\beta d_{v_{k+1}})w(x_k) \quad (5)$$

for every  $h_k \in H_k$  with  $k \in \mathbb{N}$ .

We follow the approach of Hansen and Sargent (1995) and model the preferences of the consumer recursively. For any initial income  $x_1 = x$  and  $T \in \mathbb{N}$  we define  $T$ -stage total discounted utility

$$J_T(x, \pi) := (L_{\pi_1} \circ \dots \circ L_{\pi_T})\mathbf{0}(x), \quad (6)$$

where  $\mathbf{0}$  is a function such that  $\mathbf{0}(h_k) \equiv 0$  for every  $h_k \in H_k$  and  $k \in \mathbb{N}$ . For instance, if  $T = 2$  definition (6) is read as follows

$$\begin{aligned} J_2(x, \pi) &= (L_{\pi_1} \circ L_{\pi_2})\mathbf{0}(x) = L_{\pi_1}(L_{\pi_2}\mathbf{0})(x) \\ &= u(x - \pi_1(x)) - \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma L_{\pi_2}\mathbf{0}(x, \pi_1(x), f(\pi_1(x), z))} \nu(dz) \\ &= u(x - \pi_1(x)) - \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma u(f(\pi_1(x), z) - \pi_2(x, \pi_1(x), f(\pi_1(x), z)))} \nu(dz). \end{aligned}$$

Observe that from (U1) and (P1), it follows that the sequence  $(J_T(x, \pi))_{T \in \mathbb{N}}$  is non-decreasing and bounded from below by 0 for every  $x \in \mathbb{R}_+$  and  $\pi \in \Pi$ . Moreover,

$$J_T(x, \pi) \leq \frac{dw(x)}{1 - \alpha\beta} \quad \text{for } x \in \mathbb{R}_+, \pi \in \Pi, T \in \mathbb{N}.$$

Indeed, note first that by (U2) we have

$$L_{\pi_T}\mathbf{0}(h_T) = u(x_T - \pi_T(h_T)) \leq u(x_T) \leq dw(x_T) \leq \frac{dw(x_T)}{1 - \alpha\beta}, \quad h_T \in H_T. \quad (7)$$

Now making use of (5) with  $k := T - 1$ ,  $v_T(h_T) := dw(x_T)/(1 - \alpha\beta)$ , (7) and monotonicity of the operator  $L_{\pi_{T-1}}$  we obtain

$$L_{\pi_{T-1}}(L_{\pi_T}\mathbf{0})(h_{T-1}) \leq L_{\pi_{T-1}}\left(\frac{dw}{1 - \alpha\beta}\right)(h_{T-1}) \leq dw(x_{T-1}) + \alpha\beta \frac{dw(x_{T-1})}{1 - \alpha\beta} = \frac{dw(x_{T-1})}{1 - \alpha\beta}.$$

Continuing this procedure, we finally infer that

$$\begin{aligned} J_T(x, \pi) &= (L_{\pi_1} \circ \dots \circ L_{\pi_T})\mathbf{0}(x) \leq (L_{\pi_1} \circ \dots \circ L_{\pi_{T-2}})\left(\frac{dw}{1 - \alpha\beta}\right)(x) \leq \dots \\ &\leq dw(x) + \frac{\alpha\beta dw(x)}{1 - \alpha\beta} = \frac{dw(x)}{1 - \alpha\beta}. \end{aligned}$$

By the above discussion  $\lim_{T \rightarrow \infty} J_T(x, \pi)$  exists for every  $x \in \mathbb{R}_+$  and  $\pi \in \Pi$ .

*The problem statement.* For an initial income  $x \in \mathbb{R}_+$  and policy  $\pi \in \Pi$  we define the non-expected discounted utility in the infinite time horizon as follows

$$J(x, \pi) := \lim_{T \rightarrow \infty} J_T(x, \pi). \quad (8)$$

The aim of the agent is to find an optimal value (the so-called value function) of the non-expected discounted utility in the infinite time horizon and a policy  $\pi^* \in \Pi$  for which

$$J(x, \pi^*) = \sup_{\pi \in \Pi} J(x, \pi), \quad \text{for all } x \in \mathbb{R}_+.$$

**Remark 1.** When the risk sensitive coefficient  $\gamma \rightarrow 0^+$ , then the non-expected utility in (8) tends to the von Neumann-Morgenstern expected utility that was first studied by Brock and Mirman (1972) for a stochastic optimal growth model. The greater  $\gamma > 0$  the more risk averse is the agent.

The entropic risk measure, which we used in our framework (see (3)) is also known as the exponential certainty equivalent. It can be used to study the models with robust preferences. This is because, this measure has a robust representation with the relative entropy as a penalty function, see Föllmer and Schied (2004) or Jaśkiewicz and Nowak (2011b). Moreover, as noted by Hansen and Sargent (1995) the specification of such recursion provides a bridge between risk sensitive control theory (see Whittle (1990)) and a more general recursive utility specification used by Epstein and Zin (1989).<sup>2</sup>

### 3. The Bellman Equation

In order to solve the aforementioned problem we shall use the dynamic programming approach. We start from the definition of a class of functions among which we look for a solution to the optimality equation.

For a Borel measurable function  $v : \mathbb{R}_+ \mapsto \mathbb{R}$  define its  $w$ -norm as follows

$$\|v\|_w := \sup_{x \in \mathbb{R}_+} \frac{|v(x)|}{w(x)}.$$

Let  $B_w$  be the set of all Borel measurable functions  $v : \mathbb{R}_+ \mapsto \mathbb{R}$  with the finite  $w$ -norm. Then,  $(B_w, \|\cdot\|_w)$  is a Banach space (see Proposition 7.2.1 in Hernández-Lerma and Lasserre (1999)). Define

$$\mathcal{B} := \{v \in B_w : v \text{ is continuous, concave, non-decreasing, non-negative}\}.$$

Note that  $(\mathcal{B}, \|\cdot\|_w)$  is a complete metric space as a closed subset of the Banach space  $(B_w, \|\cdot\|_w)$ .

Now we are ready to state our first result.

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<sup>2</sup>The reader is referred to Chapter 20 in Miao (2014), where a detailed discussion, further references on this topic are provided.

**Theorem 1.** Assume (U1)-(U2) and (F1)-(F2). Then, the following holds.

(a) There exist unique functions  $V \in \mathcal{B}$  and  $i^* \in \Phi$  such that

$$V(x) = \sup_{y \in [0, x]} \left( u(x - y) - \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma V(f(y, z))} \nu(dz) \right) \quad (9)$$

$$= u(x - i^*(x)) - \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma V(f(i^*(x), z))} \nu(dz) \quad (10)$$

for all  $x \in \mathbb{R}_+$ . Moreover,  $V$  is strictly concave.

(b) The functions  $x \mapsto i^*(x)$  and  $x \mapsto c^*(x) := x - i^*(x)$  are continuous and non-decreasing.

(c)  $V(x) = \sup_{\pi \in \Pi} J(x, \pi) = J(x, i^*)$  for all  $x \in \mathbb{R}_+$ , i.e. there exists an optimal stationary policy  $i^*$ .

Throughout this section we assume that (U1)-(U2) and (F1)-(F2) are satisfied. We start with a result that we shall use in many places.

**Lemma 1.** Let  $v \in \mathcal{B}$ . Then, the function

$$y \mapsto \widehat{v}(y) := -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma v(f(y, z))} \nu(dz)$$

is continuous, concave, non-decreasing and non-negative.

PROOF. By (F1) we have  $0 \leq f(0, z)$  and  $0 \leq v(0) \leq v(f(0, z))$  for any  $z \in \mathbb{R}_+$ . Hence, (P1) yields that  $0 \leq \widehat{v}$ . Furthermore, by (F1) the function  $v(f(\cdot, z))$  is non-decreasing for every  $z \in \mathbb{R}_+$ . Thus, by (P1) the function  $\widehat{v}$  is also non-decreasing. Similarly, making use again of (F1) we find that  $v(f(\cdot, z))$  is continuous for every  $z \in \mathbb{R}_+$ . Hence, the dominated convergence theorem implies that  $\widehat{v}$  is continuous. Finally, we show the concavity of  $\widehat{v}$ . Let  $y = \lambda y' + (1 - \lambda)y''$ , where  $\lambda \in (0, 1)$ . By (F1) it follows that

$$f(y, z) \geq \lambda f(y', z) + (1 - \lambda)f(y'', z), \quad z \in \mathbb{R}_+,$$

and by the fact that  $v$  is non-decreasing and concave we obtain

$$v(f(y, z)) \geq v(\lambda f(y', z) + (1 - \lambda)f(y'', z)) \geq \lambda v(f(y', z)) + (1 - \lambda)v(f(y'', z)), \quad z \in \mathbb{R}_+.$$

Now, properties (P1) and (P2) imply that

$$\widehat{v}(y) \geq -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma(\lambda v(f(y', z)) + (1 - \lambda)v(f(y'', z)))} \nu(dz) \geq \lambda \widehat{v}(y') + (1 - \lambda)\widehat{v}(y''), \quad (11)$$

which finishes the proof.  $\square$

**Lemma 2.** Assume that  $i \in \Phi$  is a non-decreasing function such that  $x \mapsto x - i(x)$  is also non-decreasing. Then, for any  $T \in \mathbb{N}$  the function  $x \mapsto J_T(x, i)$  is non-decreasing and continuous.

PROOF. Note that  $i$  is Lipschitz continuous with constant less than or equal to 1. We proceed by induction. For  $T = 1$  the assertion is true by (U1). Assume that it holds for some  $T \in \mathbb{N}$ . Then,

$$J_{T+1}(x, i) = u(x - i(x)) - \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma J_T(f(i(x), z), i)} \nu(dz). \quad (12)$$

Now the conclusion follows as in Lemma 1 with assumption (U1).  $\square$

**Lemma 3.** Assume that  $g_i$  are non-decreasing and non-negative for  $i = 1, 2$ . Then, it follows that

$$\begin{aligned} -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma(g_1(f(y, z)) + g_2(f(y, z)))} \nu(dz) &\leq \\ -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma g_1(f(y, z))} \nu(dz) - \frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma g_2(f(y, z))} \nu(dz). \end{aligned}$$

PROOF. The inequality follows from Proposition 1 in the Appendix. It suffices to define  $X := f(y, \xi)$ , where  $\xi$  is a random variable of the distribution  $\nu$  and put  $h := e^{-\gamma g_1}$ ,  $g := e^{-\gamma g_2}$ .  $\square$

For any  $v \in \mathcal{B}$ , we define the operator  $L$  as follows

$$Lv(x) := \sup_{y \in [0, x]} \left( u(x - y) - \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma v(f(y, z))} \nu(dz) \right) \quad (13)$$

for all  $x \in \mathbb{R}_+$ .

**Lemma 4.** The operator  $L$  maps  $\mathcal{B}$  into itself and is contractive.

PROOF. Let  $v \in \mathcal{B}$ . First note that by (U2) and (F2), we obtain

$$\|Lv\|_w \leq d + \alpha\beta\|v\|_w,$$

and by (U1) and Lemma 1 we have that  $Lv \geq 0$  (see also (5)). Moreover, by (U1) and Lemma 1 the function  $(x, y) \mapsto u(x - y) + \beta\widehat{v}(y)$  is continuous on  $D$ . Thus, the maximum theorem (see Berge (1963)) implies that  $Lv$  is continuous. Next, we observe that for  $x' < x''$ , we have

$$Lv(x') \leq \sup_{y \in [0, x']} (u(x'' - y) + \beta\widehat{v}(y)) \leq Lv(x'').$$

We now show that  $Lv$  is concave (see also Sec. 2.4.4 in Bäuerle and Rieder (2011)). Let  $\lambda \in (0, 1)$ ,  $x', x'' \in \mathbb{R}_+$  and  $x := \lambda x' + (1 - \lambda)x''$ . By  $y' \in [0, x']$  and  $y'' \in [0, x'']$



we denote the points that attain the maximum in (13) at  $x'$  and  $x''$ , respectively. Then,  $y := \lambda y' + (1 - \lambda)y'' \in [0, x]$ . Hence, we get

$$Lv(x) \geq u(x - y) - \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma v(f(y,z))} \nu(dz). \quad (14)$$

Moreover, by (U1) we obtain

$$u(x - y) > \lambda u(x' - y') + (1 - \lambda)u(x'' - y'). \quad (15)$$

Now combining (14) with (15) and (11) we finally obtain

$$Lv(x) > \lambda Lv(x') + (1 - \lambda)Lv(x'').$$

It only remains to prove that  $L$  is contractive. Assume that  $v_1, v_2 \in \mathcal{B}$ . Then,

$$\begin{aligned} Lv_1(x) - Lv_2(x) &\leq \sup_{y \in [0, x]} \left( -\frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma v_1(f(y,z))} \nu(dz) + \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma v_2(f(y,z))} \nu(dz) \right) \\ &\leq \beta \sup_{y \in [0, x]} \left( -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma \|v_1 - v_2\|_w w(f(y,z)) - \gamma v_2(f(y,z))} \nu(dz) + \frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma v_2(f(y,z))} \nu(dz) \right) \\ &\leq \beta \sup_{y \in [0, x]} \left( -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma \|v_1 - v_2\|_w w(f(y,z))} \nu(dz) \int_{\mathbb{R}_+} e^{-\gamma v_2(f(y,z))} \nu(dz) \right. \\ &\quad \left. + \frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma v_2(f(y,z))} \nu(dz) \right) \\ &= \beta \sup_{y \in [0, x]} -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma \|v_1 - v_2\|_w w(f(y,z))} \nu(dz) \\ &\leq \beta \sup_{y \in [0, x]} \int_{\mathbb{R}_+} \|v_1 - v_2\|_w w(f(y,z)) \nu(dz) \quad (\text{by Jensen's inequality}) \\ &\leq \alpha \beta \|v_1 - v_2\|_w(x) \quad (\text{by (F2)}). \end{aligned}$$

The second inequality follows from property (P1) and the third one from Lemma 3 ( $g_1 = \|v_1 - v_2\|_w w$ ,  $g_2 = v_2$ ) and the fact that  $w$  and  $v_2$  are non-decreasing. By changing the roles of  $v_1$  with  $v_2$  we obtain

$$\|Lv_1 - Lv_2\|_w \leq \alpha \beta \|v_1 - v_2\|_w,$$

where  $\alpha \beta < 1$ . □

**PROOF OF THEOREM 1.** Part (a) follows from Lemma 4 and the Banach fixed point theorem applied to the operator  $L$ . Hence, (9) holds. In addition, note that  $V$  is strictly concave, because of strict inequality in (15).

Since, by Lemma 1 and (U1) the function

$$y \mapsto u(x - y) - \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma V(f(y,z))} \nu(dz)$$

is continuous and strictly concave on  $[0, x]$  for  $x \in \mathbb{R}_+$ , it follows that there exists a unique point  $y^* \in [0, x]$  that realises the maximum on the right-hand side of (9). Hence, by the maximum theorem (see Berge (1963)) there exists a unique continuous function  $i^* \in \Phi$  attaining the maximum in (9). In addition, strict concavity of  $u$  and Lemma 3.2 in Balbus et al. (2015) (see also Theorem 6.3 in Topkis (1978)) imply that the function  $i^*$  is non-decreasing. Furthermore, observe that (9) can be re-formulated as follows

$$V(x) = \sup_{a \in [0, x]} \left( u(a) - \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma V(f(x-a, z))} \nu(dz) \right), \quad x \in \mathbb{R}_+.$$

Assumption (U1) and strict concavity of  $V$  and Lemma 3.2 in Balbus et al. (2015) yield that the consumption strategy  $c^* \in \Phi$  is also non-decreasing. Clearly,  $c^*(x) + i^*(x) = x$  for every  $x \in \mathbb{R}_+$ . Hence, part (b) holds true.

Part (c). From (9) it follows that

$$V(x) \geq u(x - y) - \frac{\beta}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma V(f(y, z))} \nu(dz), \quad y \in [0, x]$$

and  $x \in \mathbb{R}_+$ . Let  $\pi = (\pi_k)_{k \in \mathbb{N}} \in \Pi$  be any investment policy. Then, for any history  $h_k \in H_k$ ,  $k \in \mathbb{N}$ , the above display implies that

$$V(x_k) \geq L_{\pi_k} V(h_k). \quad (16)$$

Fix any  $T \in \mathbb{N}$ . Starting from (16) for  $k := T$  and applying (16) consecutively for  $k = T - 1, \dots, 1$  we infer that

$$V(x) \geq (L_{\pi_1} \circ \dots \circ L_{\pi_T}) V(x).$$

Since  $V \geq 0$  and  $L_{\pi_k}$  is monotone for every  $\pi_k$ ,  $k \in \mathbb{N}$ , we obtain

$$V(x) \geq (L_{\pi_1} \circ \dots \circ L_{\pi_T}) V(x) \geq (L_{\pi_1} \circ \dots \circ L_{\pi_T}) \mathbf{0}(x) = J_T(x, \pi), \quad (17)$$

for any  $\pi \in \Pi$  and  $x \in \mathbb{R}_+$ . Letting  $T \rightarrow \infty$  in (17), we finally have  $V(x) \geq J(x, \pi)$  for any  $\pi \in \Pi$  and  $x \in \mathbb{R}_+$ . Hence,

$$V(x) \geq \sup_{\pi \in \Pi} J(x, \pi) \quad x \in \mathbb{R}_+. \quad (18)$$

Let  $i^* \in \Phi$  be as in (10). For convenience of notation we set  $u_{i^*}(x) := u(x - i^*(x))$  and for any non-negative function  $\varphi \in B_w$

$$\rho_{i^*, x}(\varphi) := -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma \varphi(f(i^*(x), z))} \nu(dz), \quad L_{i^*} \varphi(x) := u_{i^*}(x) + \beta \rho_{i^*, x}(\varphi), \quad x \in \mathbb{R}_+.$$

Thus, the right hand-side of (10) equals

$$L_{i^*} V(x) = u_{i^*}(x) + \beta \rho_{i^*, x}(V), \quad x \in \mathbb{R}_+.$$

By iterating the latter equality  $T - 1$  times we get that

$$V(x) = L_{i^*}^{(T)} V(x), \quad x \in \mathbb{R}_+, \quad (19)$$

where  $L_{i^*}^{(T)}$  denotes the  $T$ -th composition of the operator  $L_{i^*}$  with itself. Thus, (19), property (P1) and Jensen's inequality together with (F2) (see also (4)) yield that

$$\begin{aligned} V(x) &= L_{i^*}^{(T-1)}(u_{i^*} + \beta \rho_{i^*, \cdot}(V))(x) \leq L_{i^*}^{(T-1)}(u_{i^*} + \beta \rho_{i^*, \cdot}(\|V\|_w w))(x) \\ &\leq L_{i^*}^{(T-1)}(u_{i^*} + \alpha \beta \|V\|_w w)(x) \\ &= L_{i^*}^{(T-2)}(u_{i^*} + \beta \rho_{i^*, \cdot}(u_{i^*} + \alpha \beta \|V\|_w w))(x) \end{aligned} \quad (20)$$

for  $x \in \mathbb{R}_+$ . Now by putting  $g_1 := u_{i^*} = J_1(\cdot, i^*)$ ,  $g_2 := \alpha \beta \|V\|_w w$  in Lemma 3, we have that

$$\begin{aligned} \beta \rho_{i^*, x'}(u_{i^*} + \alpha \beta \|V\|_w w) &\leq \beta \rho_{i^*, x'}(u_{i^*}) + \beta \rho_{i^*, x'}(\alpha \beta \|V\|_w w) \\ &\leq \beta \rho_{i^*, x'}(u_{i^*}) + (\alpha \beta)^2 \|V\|_w w(x'), \quad x' \in \mathbb{R}_+, \end{aligned} \quad (21)$$

where the second inequality is due to Jensen's inequality and assumption (F2) (see also (4)). Hence, inequalities (20) and (21) combined together get

$$\begin{aligned} V(x) &\leq L_{i^*}^{(T-2)}(u_{i^*} + \beta \rho_{i^*, \cdot}(u_{i^*}) + (\alpha \beta)^2 \|V\|_w w)(x) \\ &= L_{i^*}^{(T-3)}(u_{i^*} + \beta \rho_{i^*, \cdot}(u_{i^*} + \beta \rho_{i^*, \cdot}(u_{i^*}) + (\alpha \beta)^2 \|V\|_w w))(x). \end{aligned} \quad (22)$$

We repeat the procedure. From Lemma 2 the function

$$x \mapsto J_2(x, i^*) = u_{i^*}(x) + \beta \rho_{i^*, x}(u_{i^*})$$

is non-decreasing. Hence, making use again of Lemma 3 ( $g_1 := J_2(\cdot, i^*)$ ,  $g_2 := (\alpha \beta)^2 \|V\|_w w$ ), Jensen's inequality and (F2) we get

$$\begin{aligned} &\beta \rho_{i^*, x'}(u_{i^*} + \beta \rho_{i^*, \cdot}(u_{i^*}) + (\alpha \beta)^2 \|V\|_w w) \\ &= \beta \rho_{i^*, x'}(J_2(\cdot, i^*) + (\alpha \beta)^2 \|V\|_w w) \leq \beta \rho_{i^*, x'}(J_2(\cdot, i^*)) + \beta \rho_{i^*, x'}((\alpha \beta)^2 \|V\|_w w) \\ &\leq \beta \rho_{i^*, x'}(J_2(\cdot, i^*)) + (\alpha \beta)^3 \|V\|_w w(x'), \quad x' \in \mathbb{R}_+. \end{aligned} \quad (23)$$

By combining (22) and (23) we obtain that

$$\begin{aligned} V(x) &\leq L_{i^*}^{(T-3)}(u_{i^*} + \beta \rho_{i^*, \cdot}(J_2(\cdot, i^*)) + (\alpha \beta)^3 \|V\|_w w)(x) \\ &= L_{i^*}^{(T-3)}(J_3(\cdot, i^*) + (\alpha \beta)^3 \|V\|_w w)(x) \\ &= L_{i^*}^{(T-4)}(u_{i^*} + \beta \rho_{i^*, \cdot}(J_3(\cdot, i^*) + (\alpha \beta)^3 \|V\|_w w))(x). \end{aligned}$$

Repeating this procedure, i.e., making use of Lemma 3 for the functions  $g_1 = J_k(\cdot, i^*)$  (by Lemma 2 it is non-decreasing) and  $g_2 := (\alpha \beta)^k \|V\|_w w$  for  $k = 3, \dots, T - 1$  we finally deduce

$$V(x) \leq J_T(x, i^*) + (\alpha \beta)^T \|V\|_w w(x), \quad x \in \mathbb{R}_+. \quad (24)$$

Thus, letting  $T \rightarrow \infty$  in (24) it follows that

$$V(x) \leq J(x, i^*) \quad x \in \mathbb{R}_+. \quad (25)$$

Now, (18) and (25) combined together yield part (c).  $\square$

#### 4. The Euler Equation

This section is devoted to establish the Euler equation. Therefore, we shall need additional conditions that guarantee differentiability of functions describing the model.

(U3) The function  $u : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is continuously differentiable on  $\mathbb{R}_{++}$ .

(U4)  $u'_+(0) = \infty$ .

(F3) The function  $f(\cdot, z) : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is continuously differentiable on  $\mathbb{R}_{++}$ .

(F4)  $f(0, z) = 0$  for all  $z \geq 0$ .

(F5) There is an investment  $y > 0$  such that

$$\int_{\mathbb{R}_+} f'(y, z) \nu(dz) > 0,$$

where  $f'(y, z) := \frac{\partial f(y, z)}{\partial y}$ .<sup>3</sup>

Assumption (F5) together with (F1) rules out the trivial case that  $f(y, z) = 0$   $\nu$ -a.s. for every  $y \in \mathbb{R}_+$ .

**Theorem 2.** *Assume (U1)-(U4) and (F1)-(F5). Then, we have the following.*

(a) *For any  $x \in \mathbb{R}_{++}$  the Euler equation holds*

$$u'(c^*(x)) = \beta \frac{\int_{\mathbb{R}_+} e^{-\gamma V(f(i^*(x), z))} u'(c^*(f(i^*(x), z))) f'(i^*(x), z) \nu(dz)}{\int_{\mathbb{R}_+} e^{-\gamma V(f(i^*(x), z))} \nu(dz)}, \quad (26)$$

*where  $V$  is the function obtained in Theorem 1.*

(b) *The functions  $x \mapsto i^*(x)$  and  $x \mapsto c^*(x)$  are increasing.*

**Remark 2.** The Euler equation for the agent with risk sensitive preferences incorporates, in contrast to the standard expected utility case, the value function  $V$ . When  $\gamma = 0$ , then equation (26) becomes the well-known Euler equation for the model with the expected utility, see Brock and Mirman (1972) or Kamihigashi (2007).

Let us define

$$\widehat{V}(y) := -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma V(f(y, z))} \nu(dz). \quad (27)$$

In the subsequent lemmas we shall assume that conditions (U1)-(U4) and (F1)-(F5) hold true.

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<sup>3</sup>Note that by Theorem 7.4 in Stokey et al. (1989) it follows that the function  $z \mapsto f'(y, z)$  is Borel measurable.

**Lemma 5.** *The function  $\widehat{V}$  defined in (27) is concave, continuous and non-decreasing. Moreover,*

$$\widehat{V}'_+(0) = \infty. \quad (28)$$

PROOF. The first part follows from Lemma 1. Take any sequence  $y_n \rightarrow 0^+$  as  $n \rightarrow \infty$ . By (F4) and the fact that  $u(0) = 0$  we get that  $V(0) = 0$  and  $\widehat{V}(0) = 0$ . Hence,

$$\frac{\widehat{V}(y_n) - \widehat{V}(0)}{y_n} = -\frac{1}{\gamma y_n} \ln \int_{\mathbb{R}_+} e^{-\gamma V(f(y_n, z))} \nu(dz) \geq -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma \frac{u(f(y_n, z))}{y_n}} \nu(dz). \quad (29)$$

From the chain rule, we have for any  $z \in \mathbb{R}_+$  that

$$\frac{u(f(y_n, z))}{y_n} = \frac{u(f(y_n, z)) - u(0)}{f(y_n, z) - f(0, z)} \frac{f(y_n, z) - f(0, z)}{y_n}. \quad (30)$$

Letting  $n \rightarrow \infty$  in (30), we obtain that

$$\lim_{n \rightarrow \infty} \frac{u(f(y_n, z))}{y_n} = u'_+(f(0, z)) f'_+(0, z). \quad (31)$$

Note that the convergence in (31) is monotonic, since  $u$  and  $f(\cdot, z)$  are concave and  $f(\cdot, z)$  is non-decreasing for  $z \in \mathbb{R}_+$ . By the monotone convergence theorem, (29) and (31) we finally get

$$\widehat{V}'_+(0) \geq -\frac{1}{\gamma} \ln \int_{\mathbb{R}_+} e^{-\gamma u'_+(0) f'_+(0, z)} \nu(dz).$$

Assumption (F5) together with (F1) yield that  $f'_+(0, z) > 0$ . Thus, by (U4) the assertion follows.  $\square$

**Lemma 6.** *Let  $i^*$  be defined in (10). Then,  $i^*(x) \in (0, x)$  for any  $x \in \mathbb{R}_{++}$ .*

PROOF. The assertion that  $i^*(x) > 0$  follows from assumption (U4), whereas (28) yields  $i^*(x) < x$ . The reader is referred to Lemma 5 in Kamihigashi (2007).  $\square$

**Lemma 7.** *The function  $V$  is continuously differentiable on  $\mathbb{R}_{++}$  and  $V'(x) = u'(c^*(x))$ , for  $x \in \mathbb{R}_{++}$ .*

PROOF. The way of showing the equality proceeds along the same lines as the proofs of Proposition 12.1.18 and Corollary 12.1.19 in Stachurski (2009).  $\square$

PROOF OF THEOREM 2. First we show part (a). In view of Lemma 7 and (9), it suffices to show that  $\widehat{V}$  is differentiable on  $\mathbb{R}_{++}$  and  $\beta \widehat{V}'(y)$  at  $y = i^*(x)$  equals to the right-hand side of (26). Since  $\widehat{V}$  is concave by Lemma 5, we know that the right-hand side and the left-hand side derivatives exist. Let  $y \in \mathbb{R}_{++}$  be arbitrary and  $h > 0$ . Set

$$F(y, z) := \frac{e^{-\gamma V(f(y, z))}}{\int_{\mathbb{R}_+} e^{-\gamma V(f(y, z))} \nu(dz)} > 0$$

and note that

$$\int_{\mathbb{R}_+} F(y, z) \nu(dz) = 1 \quad \text{for any } y \in \mathbb{R}_{++}.$$

Then, by Lemma 7 and (F4)

$$\begin{aligned} \frac{\widehat{V}(y+h) - \widehat{V}(y)}{h} &= -\frac{1}{\gamma h} \ln \frac{\int_{\mathbb{R}_+} e^{-\gamma V(f(y+h, z))} \nu(dz)}{\int_{\mathbb{R}_+} e^{-\gamma V(f(y, z))} \nu(dz)} \\ &= -\frac{1}{\gamma h} \ln \int_{\mathbb{R}_+} e^{-\gamma(V(f(y+h, z)) - V(f(y, z)))} F(y, z) \nu(dz) \\ &\leq \int_{\mathbb{R}_+} \frac{V(f(y+h, z)) - V(f(y, z))}{h} F(y, z) \nu(dz) \\ &= \int_{\mathbb{R}_+} \frac{V(f(y+h, z)) - V(f(y, z))}{f(y+h, z) - f(y, z)} \frac{f(y+h, z) - f(y, z)}{h} F(y, z) \nu(dz) \\ &\leq \int_{\mathbb{R}_+} V'(f(y, z)) f'(y, z) F(y, z) \nu(dz) =: G(y), \end{aligned}$$

where the first inequality is due to Jensen's inequality and the second one follows from the fact that  $V$  and  $f(\cdot, z)$  for  $z \in \mathbb{R}_+$  are concave and non-decreasing. Thus, for  $y \in \mathbb{R}_{++}$

$$\widehat{V}'_+(y) \leq G(y). \quad (32)$$

Let us now consider the left-hand side derivative of  $\widehat{V}$ , i.e.,

$$\begin{aligned} \frac{\widehat{V}(y-h) - \widehat{V}(y)}{-h} &= \frac{1}{\gamma h} \ln \frac{\int_{\mathbb{R}_+} e^{-\gamma V(f(y-h, z))} \nu(dz)}{\int_{\mathbb{R}_+} e^{-\gamma V(f(y, z))} \nu(dz)} \\ &\geq \int_{\mathbb{R}_+} \frac{V(f(y-h, z)) - V(f(y, z))}{-h} F(y, z) \nu(dz) \\ &= \int_{\mathbb{R}_+} \frac{V(f(y-h, z)) - V(f(y, z))}{f(y-h, z) - f(y, z)} \frac{f(y-h, z) - f(y, z)}{-h} F(y, z) \nu(dz) \\ &\geq \int_{\mathbb{R}_+} V'(f(y, z)) f'(y, z) F(y, z) \nu(dz) = G(y). \end{aligned}$$

Hence, for  $y \in \mathbb{R}_{++}$

$$\widehat{V}'_-(y) \geq G(y). \quad (33)$$

Observe that  $G$  is continuous. This is due to Lemma 7, (F1), (F4) and the dominated convergence theorem. Since  $\widehat{V}$  is concave and (32) and (33) hold, then for  $h > 0$  we obtain that

$$G(y+h) \leq \widehat{V}'_-(y+h) \leq \widehat{V}'_+(y) \leq G(y) \leq \widehat{V}'_-(y) \leq \widehat{V}'_+(y-h) \leq G(y-h). \quad (34)$$

Now letting  $h \rightarrow 0^+$  in (34), it follows that  $\widehat{V}$  is continuously differentiable on  $\mathbb{R}_{++}$ .

In order to prove (b) suppose that  $x' < x''$ . If  $x' = 0$ , then by Lemma 6 we have  $i^*(x') = 0 < i^*(x'')$  and  $c^*(x') = 0 < c^*(x'') = x'' - i^*(x'')$ . Hence, let  $x' > 0$  and  $i^*(x') = i^*(x'')$ . Then, by Euler equation (26), we obtain  $u'(x' - i^*(x')) = u'(x'' - i^*(x'))$ . But the equality cannot hold, since  $u$  is strictly concave. Similarly, if  $c^*(x') = c^*(x'')$ , then by Lemma 7 we must have  $V'(x') = u'(c^*(x')) = u'(c^*(x'')) = V'(x'')$ . However, this equality contradicts the strict concavity of  $V$ .  $\square$

## 5. Stationary Distributions

In this section, we shall consider dynamics of the growth model when the agent follows the optimal policy  $i^* \in \Phi$ . Our aim is to provide a set of assumptions under which the system is globally stable and the resulting stationary distribution is non-trivial in the sense that it is not concentrated on zero. We wish to follow the approach studied in Nishimura and Stachurski (2005) and further developed by Kamihigashi (2007) and Stachurski (2009). It combines the Euler equation (see (26)) with the Foster-Lyapunov theory of Markov chains.

More precisely, we deal with the process

$$x_{t+1} = f(i^*(x_t), \xi_t), \quad (\xi_t)_{t \in \mathbb{N}} \text{ is i.i.d. sequence, where } \xi_t \sim \nu \in \text{Pr}(\mathbb{R}_+) \quad (35)$$

and  $f : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$  is continuous. Clearly,  $(x_t)_{t \in \mathbb{N}}$  is a Markov process. Assuming that  $f(y, z) > 0$  for every  $y \in \mathbb{R}_{++}$  and utilising the fact that  $i^*(x) \in (0, x)$  for  $x > 0$ , we may confine ourselves to the study of the income process on  $\mathbb{R}_{++}$  (see p. 303 in Stachurski (2009)). We show the existence of at least one stationary non-trivial distribution. According to Proposition 2 in the Appendix, we have to find a function  $W : \mathbb{R}_{++} \mapsto \mathbb{R}_+$  satisfying properties (a) and (b). Since we wish to avoid repeating all the details contained in the aforementioned papers, we focus only on an element in the proof that makes use of the Euler equation. This is because, the Euler equation in our framework is different than the one obtained for the expected utility case. Therefore, we formulate only crucial conditions in the most simple way. The reader is referred to Kamihigashi (2007) and Stachurski (2009), where a detailed discussion and more general conditions are provided implying the ones given below.

(D1) It is satisfied that

$$\lim_{y \rightarrow 0^+} \int_{\mathbb{R}_+} \frac{1}{\beta f'(y, z)} \nu(dz) < 1.$$

(D2) There exist  $\lambda_2 \in (0, 1)$  and  $\kappa_2 > 0$  such that

$$\int_{\mathbb{R}_+} f(y, z) \nu(dz) \leq \lambda_2 y + \kappa_2, \quad y \in \mathbb{R}_+.$$

Here, we would like to mention that Assumption (D1) prevents probability mass from escaping to infinity, whereas the role of (D2) is to prevent probability mass from escaping to zero.

**Lemma 8.** Assume (D1). Then, for  $W_1(x) := \sqrt{u'(c^*(x))e^{-\gamma V(x)}}$ ,  $x \in \mathbb{R}_{++}$ , there exist  $\lambda_1 \in (0, 1)$  and  $\kappa_1 > 0$  such that

$$\int_{\mathbb{R}_+} W_1(f(i^*(x), z))\nu(dz) \leq \lambda_1 W_1(x) + \kappa_1, \quad x \in \mathbb{R}_{++}.$$

PROOF. First note that by the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \int_{\mathbb{R}_+} W_1(f(i^*(x), z))\nu(dz) &= \int_{\mathbb{R}_+} \left[ u'(c^*(f(i^*(x), z)))e^{-\gamma V(f(i^*(x), z))} \frac{\beta f'(i^*(x), z)}{\beta f'(i^*(x), z)} \frac{\int_{\mathbb{R}_+} e^{-\gamma V(f(i^*(x), z))}\nu(dz)}{\int_{\mathbb{R}_+} e^{-\gamma V(f(i^*(x), z))}\nu(dz)} \right]^{\frac{1}{2}} \nu(dz) \\ &\leq \left[ \int_{\mathbb{R}_+} u'(c^*(f(i^*(x), z)))e^{-\gamma V(f(i^*(x), z))} \frac{\beta f'(i^*(x), z)}{\int_{\mathbb{R}_+} e^{-\gamma V(f(i^*(x), z))}\nu(dz)} \nu(dz) \right]^{\frac{1}{2}} \times \\ &\quad \times \left[ \int_{\mathbb{R}_+} \frac{\int_{\mathbb{R}_+} e^{-\gamma V(f(i^*(x), z))}\nu(dz)}{\beta f'(i^*(x), z)} \nu(dz) \right]^{\frac{1}{2}}, \quad x \in \mathbb{R}_{++}. \end{aligned} \quad (36)$$

Furthermore, applying (26) to (36) we obtain that

$$\begin{aligned} \int_{\mathbb{R}_+} W_1(f(i^*(x), z))\nu(dz) &\leq \sqrt{u'(c^*(x))} \left[ \int_{\mathbb{R}_+} \frac{\nu(dz)}{\beta f'(i^*(x), z)} \int_{\mathbb{R}_+} e^{-\gamma V(f(i^*(x), z))}\nu(dz) \right]^{\frac{1}{2}} \\ &\leq \sqrt{u'(c^*(x))} \left[ \int_{\mathbb{R}_+} \frac{\nu(dz)}{\beta f'(x, z)} \right]^{\frac{1}{2}}, \end{aligned} \quad (37)$$

where the second inequality is due to the fact that  $V \geq 0$  and (F1) ( $f'(\cdot, z)$  is non-increasing for  $z \in \mathbb{R}_+$ ). From assumption (D1), it follows that there exists  $\delta > 0$  such that

$$\lambda_1 := e^{\gamma/2V(\delta)} \left[ \int_{\mathbb{R}_+} \frac{\nu(dz)}{\beta f'(\delta, z)} \right]^{\frac{1}{2}} < 1.$$

Then, for  $x \in (0, \delta)$  we have that  $e^{-\gamma/2V(x)}e^{\gamma/2V(\delta)} \geq 1$ , and consequently, by (37)

$$\int_{\mathbb{R}_+} W_1(f(i^*(x), z))\nu(dz) \leq \sqrt{u'(c^*(x))e^{-\gamma V(x)}e^{\gamma/2V(\delta)}} \left[ \int_{\mathbb{R}_+} \frac{\nu(dz)}{\beta f'(\delta, z)} \right]^{\frac{1}{2}} \leq \lambda_1 W_1(x). \quad (38)$$

For  $x \geq \delta$  we have

$$\int_{\mathbb{R}_+} W_1(f(i^*(x), z))\nu(dz) \leq \int_{\mathbb{R}_+} [u'(c^*(f(i^*(\delta), z)))e^{-\gamma V(f(i^*(\delta), z))}]^{\frac{1}{2}} \nu(dz) =: \kappa_1. \quad (39)$$

Inequalities (38) and (39) combined together yield the conclusion.  $\square$



From Lemma 8 and (D2) we deduce that the function

$$W(x) = W_1(x) + x, \quad x \in \mathbb{R}_{++}$$

satisfies point (b) in Proposition 2 with  $\lambda := \max\{\lambda_1, \lambda_2\}$  and  $\kappa := \kappa_1 + \kappa_2$ . Clearly, from (U4) it follows that condition (a) is also satisfied. Hence, there exists a non-trivial distribution for the Markov process in (35).

The problem of the uniqueness of a stationary distribution, i.e., global stability, has been recently studied in Kamihigashi (2007), Nishimura and Stachurski (2005) and Stachurski (2009). Therefore, we do not give here all these assumptions and refer the reader to the above-mentioned works. Further results on invariant distributions are also widely reported in Meyn and Tweedie (2009), Bhattacharya and Majumdar (2007) and in the recent papers of Kamihigashi and Stachurski (2014), Zhang (2007).

**Remark 3.** If the income process evolves on the compact space  $[0, \bar{s}]$ , then the assumptions (U2) and (F2) are satisfied with  $w \equiv 1$ . Consequently, the results in Sections 3 and 4 are satisfied, in particular, the optimal investment policy is non-decreasing. In this case, the existence of a non-trivial invariant distribution follows from the Krylov-Bogolubov theorem, see for instance, Theorem 11.2.5 in Stachurski (2009).

## 6. Examples

Below we provide two examples of utility and production functions that meet our assumptions used in Sections 3-4.

**Example 1.** *A model with multiplicative shocks.* Assume that the process evolves according to difference equation

$$x_{t+1} = y_t^\theta \xi_t, \quad t \in \mathbb{N},$$

where  $\theta \in (0, 1)$ . Suppose that  $\bar{z} := \int_{\mathbb{R}_+} z \nu(dz)$  is finite and let  $u(a) = a^\sigma$  with  $\sigma \in (0, 1)$ . Clearly, (U1) and (F1) are satisfied. Assumption (U2) holds for  $w(x) = (r + x)^\sigma$ , where  $r \geq 1$  is a constant sufficiently large so that

$$\left(1 + \frac{\bar{z}^{\frac{1}{1-\theta}}}{r}\right)^\sigma \beta < 1.$$

Then, calculations on p. 263 in Jaśkiewicz and Nowak (2011b) show that (F2) is also satisfied with  $\alpha := \left(1 + \frac{\bar{z}^{1/(1-\theta)}}{r}\right)^\sigma$ . We also note that (U3)-(U4) and (F3)-(F5) hold true.

For this model, conditions (D1) and (D2) are met as well. Clearly, the finiteness of  $\int_{\mathbb{R}_+} 1/z \nu(dz)$  implies (D1). Define now

$$\kappa_1 := \max\{\bar{z}, \bar{z}^{\frac{1}{1-\theta}}(1-\theta)\}, \quad \lambda_1 := \theta.$$

Then, for  $y \leq 1$  we have

$$\int_{\mathbb{R}_+} y^\theta z \nu(dz) = y^\theta \bar{z} \leq \bar{z} \leq \theta y + \kappa_1.$$

Set  $l(y) := \theta y - \bar{z} y^\theta + \kappa_1$  for  $y > 0$ . This function attains minimum at

$$y_{\min} = \bar{z} y^{\frac{1}{1-\theta}} \quad \text{and} \quad l(y_{\min}) \geq 0.$$

Hence,

$$\int_{\mathbb{R}_+} f(y, z) \nu(dz) = \bar{z} y^\theta \leq \lambda_1 y + \kappa_1$$

for all  $y \in \mathbb{R}_+$ .

**Example 2.** *A model with additive shocks.* Let the evolution of the process be described by the equation

$$x_{t+1} = \begin{cases} \eta y_t + \xi_t, & \text{if } y_t > 0, \\ 0, & \text{if } y_t = 0, \end{cases} \quad t \in \mathbb{N},$$

where  $\eta > 0$  denotes a constant rate of growth. Obviously, (F1), (F3)-(F5) holds. Let the utility  $u$  be defined as in Example 1. Then, (U2) holds for  $w(x) = (x + r)^\sigma$  with any  $r \geq 1$ . Now, we prove assumption (F2). Namely, by the Jensen inequality it follows that

$$\sup_{y \in [0, x]} \int_{\mathbb{R}_+} (\eta y + z) \nu(dz) \leq (\eta x + \bar{z} + r)^\sigma.$$

Thus,

$$\sup_{x \in \mathbb{R}_+} \frac{(\eta x + \bar{z} + r)^\sigma}{w(x)} = (s(x))^\sigma, \quad \text{where} \quad s(x) := \frac{\eta x + \bar{z} + r}{x + r}, \quad x \in \mathbb{R}_+.$$

We have

$$\lim_{x \rightarrow 0^+} s(x) = 1 + \frac{\bar{z}}{r}, \quad \lim_{x \rightarrow +\infty} s(x) = \eta.$$

If  $\eta > 1$ , then from the calculations in Jaśkiewicz and Nowak (2011b) on p. 264, it follows that for  $r > \max\{1, \frac{\bar{z}}{\eta-1}\}$  condition (F2) holds for all  $\beta \in (0, 1)$  for which  $\beta \eta^\sigma < 1$  (here  $\alpha := \eta^\sigma$ ). If, on the other hand,  $\eta \leq 1$ , then (F2) is satisfied for each  $\beta \in (0, 1)$ . Namely, for the given discount factor it is enough to take sufficiently large  $r \geq 1$  such that  $(1 + \frac{\bar{z}}{r})^\sigma \beta < 1$ . Obviously,  $\alpha := (1 + \frac{\bar{z}}{r})^\sigma$ .

In this case, assumption (D1) is not met. However, the existence of an invariant distribution can be proved under some extra requirements with the help of other techniques, see for instance, p. 207 and p. 259 in Stachurski (2009) or Meyn and Tweedie (2009).

## 7. Appendix

The following result can be found in Devroye et al. (1996) (Theorem A.19).

**Proposition 1.** *Let  $X$  be a real-valued random variable defined on  $(\Omega, \mathcal{F}, P)$  and let  $h$  and  $g$  be non-increasing real-valued functions. Then,*

$$E\{h(X)g(X)\} \geq E\{h(X)\}E\{g(X)\},$$

*provided that all expectations exist and are finite.*

For the proof of the next result the reader is referred to Kamihigashi (2007) (Lemma 3.1) or to Stachurski (2009) (Corollary 11.2.10).

**Proposition 2.** *Consider the Markov process defined in (35). Suppose that there exists a function  $W : \mathbb{R}_{++} \mapsto \mathbb{R}_+$  such that*

$$(a) \quad \lim_{x \rightarrow \infty} W(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^+} W(x) = \infty$$

$$(b) \quad \exists \kappa > 0, \exists \lambda \in [0, 1), \forall x \in \mathbb{R}_{++} \quad \int_{\mathbb{R}_+} W(f(i^*(x), z)) \nu(dz) \leq \lambda W(x) + \kappa.$$

*Then, there exists at least one non-trivial stationary distribution on  $\mathbb{R}_+$ .*

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